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LU Decomposition Method

LU decomposition method relies on the principle that every square matrix 'A' can be expresed in the form of LU where L is a unit lower triangular matrix and U is upper triangular matrix. provided A is non-singular.

i.e., if $A = [a_{ij}]_{n \times n}$

then

$$a_{11} \neq 0, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0$$

Suppose a ~~co~~ system of equations is given by

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

then $AX = B$ —— (1)

where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Let $A = LU$

where $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}, U = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$ (2)

then from (1) & (2) we get

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \cdot \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ l_{21}U_{11} & l_{21}U_{12}+U_{22} & l_{21}U_{13}+U_{23} \\ l_{31}U_{11} & l_{31}U_{12}+l_{32}U_{22} & (l_{31}U_{13}+l_{32}U_{23}+U_{33}) \end{bmatrix}$$

Now compairing the two matrices
we get $a_{11} = U_{11}$, $a_{12} = U_{12}$, $a_{13} = U_{13}$

$$l_{21}U_{11} = a_{21} \Rightarrow l_{21} = \frac{a_{21}}{U_{11}} = \frac{a_{21}}{a_{11}}$$

$$l_{31}U_{11} = a_{31} \Rightarrow l_{31} = \frac{a_{31}}{a_{11}}$$

$$l_{21}U_{12} + U_{22} = a_{22} \Rightarrow l_{21} = \frac{a_{22} - U_{22}}{U_{12}}$$

$$\Rightarrow U_{22} = a_{22} - l_{21}U_{12} = a_{22} - \frac{a_{21} \cdot a_{12}}{a_{11}}$$

$$l_{21}U_{13} + U_{23} = a_{23}$$

$$\Rightarrow U_{23} = a_{23} - \frac{a_{21} \cdot a_{13}}{a_{11}}$$

$$l_{31}U_{12} + l_{32}U_{22} = a_{32}$$

$$\Rightarrow l_{32} = \left[a_{32} - \frac{a_{31}a_{12}}{a_{11}}\right] \Big/ \left[a_{22} - \frac{a_{21}a_{12}}{a_{11}}\right]$$

$$l_{31}U_{13} + l_{32}U_{23} + U_{33} = a_{33}$$

$$\Rightarrow U_{33} = a_{33} - \frac{a_{31}a_{13}}{a_{11}} - \frac{\left[a_{32} - \frac{a_{31}a_{12}}{a_{11}}\right]\left[a_{23} - \frac{a_{21} \cdot a_{13}}{a_{11}}\right]}{\left[a_{22} - \frac{a_{21} \cdot a_{12}}{a_{11}}\right]}$$

From the above values we can derive the elements of L and U. Replacing A by LU we get $LUX = B$ —— (3)

$UX = Y$ (let) —— (4)

where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

From (3) and (4) we get

$$LY = B$$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

from this equation we get

$$\left.\begin{aligned} y_1 &= b_1 \\ y_2 &= b_2 - l_{21} y_1 \\ y_3 &= b_3 - l_{31} b_1 - l_{32} b_2 \end{aligned}\right\} \text{—— (5)}$$

Now from eqn (4) we get

$$\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

From this eqn we get

$$x_3 = (y_3 / U_{33})$$

$$x_2 = \left(\frac{y_2 - U_{23} x_3}{U_{22}}\right)$$

$$x_1 = \frac{y_1 - U_{12} x_2 - U_{13} x_3}{U_{11}}$$

Ques Solve the following by LU-Decomposition Method.

$$2x_1 + x_2 + x_3 = 2 \quad \text{--- (1)}$$
$$x_1 + 3x_2 + 2x_3 = 2 \quad \text{--- (2)}$$
$$3x_1 + x_2 + 2x_3 = 2. \quad \text{--- (3)}$$

Soln Here the co-efficient matrix is given by $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ 3 & 1 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $B = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$

$\because$ $A = LU$

$$\Rightarrow \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

On comparing matrices we get

$$u_{11} = 2, \quad u_{12} = 1, \quad u_{13} = 1$$

$l_{21} u_{11} = 1 \Rightarrow l_{21} = 1/2$

$l_{31} = 3/2$

$u_{22} = 3 - \left(\frac{1}{2}\right)1 = 5/2$

$u_{23} = 2 - \left(\frac{1}{2}\right)1 = 3/2$

$$l_{32} = \frac{1 - (3/2)1}{5/2} = -1/5$$

$$u_{33} = 2 - \frac{3}{2}(1) - \left(\frac{-1}{5}\right)\left(\frac{3}{2}\right) = \frac{4}{5}$$

$$\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -1/5 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 5/2 & 3/2 \\ 0 & 0 & 4/5 \end{bmatrix}$$

$\because LY = B$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -1/5 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

$\Rightarrow y_1 = 2$

$\frac{1}{2} \cdot y_1 + y_2 = 2 \Rightarrow y_2 = 2 - 1 = 1$

$\frac{3}{2} y_1 - \frac{1}{5} y_2 + y_3 = 2 \Rightarrow 3 - \frac{1}{5} + y_3 = 2$

$\Rightarrow y_3 = (-4/5)$

$\because$ We have $UX = Y$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 5/2 & 3/2 \\ 0 & 0 & 4/5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -4/5 \end{bmatrix}$$

$\Rightarrow 2x_1 + x_2 + x_3 = 2$

$\frac{5}{2} x_2 + \frac{3}{2} x_3 = 1$

$\frac{4}{5} x_3 = -4/5$

$\Rightarrow x_3 = -1$

$\therefore \frac{5}{2} x_2 + \frac{3}{2}(-1) = 1 \Rightarrow x_2 = 1$

$\Rightarrow 2x_1 + 1 - 1 = 2 \Rightarrow x_1 = 1$

Ans